Note 1. The factor $e^{\int P\,dx}$, on multiplying by which the LHS of the differential equation becomes the differential co-efficient of some function of x and y, is called an integrating factor of the differential equation and is shortly written as I.F.

Note 2. The solution of the linear equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only, is

$$y(\text{I.F.}) = \int Q(\text{I.F.})\, dx + c$$

Note 3. Sometimes a differential equation becomes linear if we take y as the independent variable and x as dependent variable. In that case, the equation can be put in the form $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y (and not of x) or constants.

I.F. (in this case) $= e^{\int P\,dy}$, and the solution is

$$x(\text{I.F.}) = \int Q.\,(\text{I.F.})\, dy + c.$$

Note 4. While evaluating the I.F., it is very useful to remember that $e^{\log f(x)} = f(x)$.

Thus, $e^{\log x^2} = x^2$.

Note 5. The co-efficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing throughout by it.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following :*

(i) $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ (ii) $\frac{dy}{dx} = y \tan x - 2 \sin x$.

Sol. (i) Given equation is $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Dividing throughout by $1 + x^2$, $\left(\text{to make the co-efficient of } \frac{dy}{dx} \text{ unity.}\right)$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{4x^2}{1+x^2} \quad \ldots(i)$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here, $P = \frac{2x}{1+x^2}, Q = \frac{4x^2}{1+x^2}$

$\therefore$ $\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\, dx + c$$

or

$$y(1 + x^2) = \int \frac{4x^2}{1+x^2}\,.\,(1 + x^2)\, dx + c$$

or $$y(1+x^2) = \int 4x^2 dx + c$$

or $$y(1+x^2) = \frac{4x^3}{3} + c.$$

(*ii*) Given equation is $\frac{dy}{dx} - (\tan x) \cdot y = -2 \sin x$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $$P = -\tan x,\ Q = -2 \sin x$$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{-\int \tan x\,dx} = e^{-(-\log \cos x)}$$
$$= e^{\log \cos x} = \cos x$$

Hence the solution is

$$y\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\,dx + c$$

or $$y \cos x = \int -2 \sin x \cos x\,dx + c$$
$$= -\int \sin 2x\,dx + c = -\frac{-\cos 2x}{2} + c$$

or $$y \cos x = \tfrac{1}{2} \cos 2x + c.$$

Example 2. *Solve the following*:

(*i*) $\sec x \frac{dy}{dx} = y + \sin x$ (*ii*) $x \log x \frac{dy}{dx} + y = 2 \log x$

Sol. (*i*) Given equation is $\sec x \cdot \frac{dy}{dx} - y = \sin x$

Dividing throughout by $\sec x$, to make the co-efficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} - (\cos x) \cdot y = \sin x \cos x$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here, $$P = -\cos x,\quad Q = \sin x \cos x$$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{\int -\cos x\,dx} = e^{-\sin x}$$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\,dx + c$$

or $$y\,.\,e^{-\sin x} = \int \sin x \cos x\,.\,e^{-\sin x}\,dx + c = \int te^{-t}\,dt + c, \text{ where } t = \sin x$$
$$= t\,.\,\frac{e^{-t}}{-1} - \int 1\,.\,\frac{e^{-t}}{-1}\,dt + c = -te^{-t} - e^{-t} + c$$
$$= -e^{-t}(t+1) + c = -e^{-\sin x}(\sin x + 1) + c$$

or $$y = -(\sin x + 1) + c\,e^{\sin x}.$$

(*ii*) Given equation is $x \log x \dfrac{dy}{dx} + y = 2 \log x$

Dividing throughout by $x \log x$ to make the co-efficient of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{1}{x \log x} \cdot y = \frac{2}{x}.$$

It is of the form $\quad \dfrac{dy}{dx} + Py = Q$

Here, $\quad P = \dfrac{1}{x \log x}, \quad Q = \dfrac{2}{x}.$

$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1/x}{\log x} dx} = e^{\log \log x} = \log x$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int Q.\,(\text{I.F.})\,dx + c$$

or $\quad y \log x = \displaystyle\int \frac{2}{x} \log x \, dx + c$

or $\quad y \log x = 2 \displaystyle\int \frac{1}{x} \cdot \log x \, dx + c = 2 \cdot \frac{(\log x)^2}{2} + c \quad \left| \because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1 \right.$

or $\quad y \log x = (\log x)^2 + c.$

Example 3. *Solve:* $\quad x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1).$

Sol. Given equation is

$$x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$$

or $$\frac{dy}{dx} - \frac{x-2}{x(x-1)}\,y = \frac{x^2(2x-1)}{x-1}$$

It is of the form $\quad \dfrac{dy}{dx} + Py = Q$

Here, $\quad P = -\dfrac{x-2}{x(x-1)}, \quad Q = \dfrac{x^2(2x-1)}{x-1}$

$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{x-2}{x(x-1)} dx} = e^{-\int \left(\frac{2}{x} - \frac{1}{x-1}\right) dx}$

$$= e^{-[2 \log x - \log(x-1)]} = e^{-[\log x^2 - \log(x-1)]}$$

$$= e^{-\log \frac{x^2}{x-1}} = e^{\log \left(\frac{x^2}{x-1}\right)^{-1}} = \left(\frac{x^2}{x-1}\right)^{-1} = \frac{x-1}{x^2}$$

$\therefore$ The solution is

$$y \cdot \frac{x-1}{x^2} = \int \frac{x^2(2x-1)}{x-1} \cdot \frac{x-1}{x^2}\, dx + c = \int (2x-1)\, dx + c = x^2 - x + c$$

or $$y(x-1) = x^2(x^2 - x + c).$$

Example 4. *Solve:* $x(1-x^2)\dfrac{dy}{dx} + (2x^2-1)y = x^3.$

Sol. Dividing by $x(1-x^2)$ to make the co-efficient of $\dfrac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y = \frac{x^2}{1-x^2}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $$P = \frac{2x^2-1}{x(1-x^2)},\quad Q = \frac{x^2}{1-x^2}$$

Now $$P = \frac{2x^2-1}{x(1-x)(1+x)} = -\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

[Partial fractions]

$$\therefore \quad \int P dx = -\log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$

$$= -\log[x(1-x)^{1/2}(1+x)^{1/2}]$$

$$= -\log[x\sqrt{1-x^2}] = \log(x\sqrt{1-x^2})^{-1}$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\log(x\sqrt{1-x^2})^{-1}} = \frac{1}{x\sqrt{1-x^2}}$$

The solution is

$$y \cdot \frac{1}{x\sqrt{1-x^2}} = \int \frac{x^2}{1-x^2} \cdot \frac{1}{x\sqrt{1-x^2}}\, dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}\, dx + c = -\frac{1}{2}\int (1-x^2)^{-3/2} \cdot (-2x)\, dx + c$$

$$= -\frac{1}{2} \cdot \frac{(1-x^2)^{-1/2}}{-\frac{1}{2}} + c$$

$$\Rightarrow \quad \frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c \quad \Rightarrow \quad y = x + cx\sqrt{1-x^2}$$

which is the required solution.

Example 5. *Solve:* $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1.$

Sol. The given equation is

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

or $$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}}y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is of the form $$\frac{dy}{dx} + Py = Q$$

Here $$P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{\int x^{-1/2} dx} = e^{2\sqrt{x}}$$

$\therefore$ Hence the solution is

$$y \,.\, e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \,.\, e^{2\sqrt{x}}\, dx + c$$

or $$y \,.\, e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + c$$

or $$ye^{2\sqrt{x}} = 2\sqrt{x} + c \quad \text{or} \quad y = e^{-2\sqrt{x}}(2\sqrt{x} + c).$$

Equations of the Form $\dfrac{dx}{dy} + Px = Q$ where P and Q are functions of y only.

Example 6. *Solve the following:*

(i) $(1 + y^2) + (x - e^{\tan^{-1} y})\dfrac{dy}{dx} = 0$ *(ii)* $(2x - 10y^3)\dfrac{dy}{dx} + y = 0.$

Sol. *(i)* The given equation is

$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

or $$(1 + y^2)\frac{dx}{dy} + x - e^{\tan^{-1} y} = 0$$

or $$\frac{dx}{dy} + \frac{1}{1+y^2} \,.\, x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

It is of the form $$\frac{dx}{dy} + Px = Q$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is

$$x.\, e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + c = \int e^t \cdot e^t\, dt + c \quad \text{where } t = \tan^{-1} y$$

$$= \int e^{2t}\, dt + c = \tfrac{1}{2} e^{2t} + c$$

or $$x \,.\, e^{\tan^{-1} y} = \tfrac{1}{2} e^{2\tan^{-1} y} + c.$$

(*ii*) The given equation is $(2x - 10y^3) \dfrac{dy}{dx} + y = 0$

or $$y \cdot \frac{dx}{dy} + 2x - 10y^3 = 0 \quad \text{or} \quad \frac{dx}{dy} + \frac{2}{y} \cdot x = 10y^2$$

It is of the form $\dfrac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int P\, dx} = e^{\int \frac{2}{y} dy} = e^{2 \log y} = e^{\log y^2} = y^2$$

$\therefore$ The solution is $$xy^2 = \int 10y^2 \,.\, y^2 dy + c = 10 \int y^4\, dy + c$$

or $$xy^2 = \frac{10y^5}{5} + c = 2y^5 + c.$$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$
2. $\dfrac{dy}{dx} + y \sec x = \tan x$
3. $\dfrac{dy}{dx} + y \tan x = \sec x$
4. $(1 + x^2) \dfrac{dy}{dx} + 2xy = \cos x$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{x + 1}$
6. $(x + 1) \dfrac{dy}{dx} - ny = e^x (x + 1)^{n+1}$
7. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$
8. $(1 + x^2) \dfrac{dy}{dx} + y = \tan^{-1} x$
9. $\dfrac{dy}{dx} + \dfrac{2x}{x^2 + 1} \cdot y = \dfrac{1}{(x^2 + 1)^2}$ given that $y = 0$ when $x = 1$
10. $\dfrac{dy}{dx} + 2y \tan x = \sin x$ given that $y = 0$ when $x = \dfrac{\pi}{3}$
11. $x\dfrac{dy}{dx} + 2y = x^2 \log x$
12. $\dfrac{dy}{dx} + y \cos x = \sin 2x$
13. $\dfrac{dy}{dx} = x(x^2 - 2y)$
14. $\sin x \dfrac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$
15. $(1 - x^2) \dfrac{dy}{dx} + xy = ax$
16. $x(x - 1) \dfrac{dy}{dx} - y = x^2(x - 1)^2$

17. $y\,dx - x\,dy + \log x\,dx = 0$

18. $\dfrac{dy}{dx} + 2y\cot x = 3x^2 \operatorname{cosec}^2 x$

19. $\sin 2x \dfrac{dy}{dx} = y + \tan x$

20. $(x + 2y^3)\dfrac{dy}{dx} = y$

21. $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

22. $e^y\,dx + (1 + xe^y)\,dy = 0$

23. $\dfrac{dx}{dy} + 2x = 6e^y$

24. $2y' + 4y = x^2 - x$

25. $y' - 2y = \cos 3x$

26. $\dfrac{dy}{dx} + y\cot x = 2x + x^2 \cot x$

27. $y' + y = \dfrac{1 + x\log x}{x}$

28. $xy' - y = (x - 1)\,e^x$

Answers

1. $xy = \frac{1}{4}x^4 + c$

2. $y(\sec x + \tan x) = \sec x + \tan x - x + c$

3. $y = \sin x + c\cos x$

4. $y(1 + x^2) = \sin x + c$

5. $\dfrac{y}{x+1} = \log(x + 1) + c$

6. $y = (x + 1)^n (e^x + c)$

7. $y = \tan x - 1 + ce^{-\tan x}$

8. $y = \tan^{-1} x - 1 + ce^{-\tan x}$

9. $y(x^2 + 1) = \tan^{-1} x - \dfrac{\pi}{4}$

10. $y = \cos x - 2\cos^2 x$

11. $x^2 y = \dfrac{x^4}{4}\log x - \dfrac{x^4}{16} + c$

12. $y = 2(\sin x - 1) + ce^{-\sin x}$

13. $y = \dfrac{1}{2}(x^2 - 1) + ce^{-x^2}$

14. $y\sin x = \dfrac{2}{3}\sin^3 x + c$

15. $y = a + c\sqrt{1 - x^2}$

16. $y = \left(1 - \dfrac{1}{x}\right)\left(\dfrac{x^3}{3} + c\right)$

17. $y + 1 + \log x = cx$

18. $y\sin^2 x = x^3 + c$

19. $y = \tan x + c\sqrt{\tan x}$

20. $x = y^3 + cy$

21. $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$

22. $xe^y + y = c$

23. $x = 2e^y + ce^{-2y}$

24. $y = \dfrac{1}{4}(x - 1)^2 + ce^{-2x}$

25. $y = \dfrac{1}{13}(3\sin 3x - 2\cos 3x) + ce^{2x}$

26. $y\sin x = x^2 \sin x + c$

27. $y = \log x + ce^{-x}$

28. $y = e^x + cx$

BERNOULLI'S EQUATION
(Equations Reducible to the Linear Form)

2.3 TO SOLVE THE EQUATION $\frac{dy}{dx} + Py = Qy^n$, WHERE P AND Q ARE FUNCTIONS OF x ONLY

The given equation is $\frac{dy}{dx} + Py = Qy^n$ ...(*i*)

Dividing both sides of (*i*) by y^n, to make the RHS a function of x only.

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad ...(ii)$$

Put $y^{1-n} = z$, then

$$(1-n)\,.\,y^{-n}\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\,.\,\frac{dz}{dx}$$

$\therefore$ (*ii*) becomes $\frac{1}{1-n}\,.\,\frac{dz}{dx} + Pz = Q$

or $$\frac{dz}{dx} + (1-n)\,.\,Pz = (1-n)\,Q.$$

which is a linear equation in z and can be solved.

In the solution, putting $z = y^{1-n}$, we get the required solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$.

Sol. The given equation is $2\,.\,\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$.

Dividing throughout by y^2

$$2y^{-2}\frac{dy}{dx} - \frac{1}{x}\,.\,y^{-1} = \frac{1}{x^2} \qquad ...(i)$$

Put $y^{-1} = z$, then $-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes

$$-2\frac{dz}{dx} - \frac{1}{x}z = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} + \frac{1}{2x}z = -\frac{1}{2x^2}$$

which is linear in z.

$$P = \frac{1}{2x},\ Q = -\frac{1}{2x^2}$$

$$\text{I.F.} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\log x} = e^{\log\sqrt{x}} = \sqrt{x}$$

$\therefore$ The solution is $z \cdot \sqrt{x} = \int -\frac{1}{2x^2} \sqrt{x}\, dx + c$

or $$y^{-1} \sqrt{x} = -\frac{1}{2} \int x^{-3/2}\, dx + c \quad \text{or} \quad \frac{\sqrt{x}}{y} = \frac{1}{\sqrt{x}} + c$$

or $$x = y(1 + c\sqrt{x}).$$

Example 2. *Solve the following :*

(i) $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ (ii) $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$.

Sol. (i) The given equation is $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing throughout by e^y

$$e^{-y} \frac{dy}{dx} + e^{-y} \frac{dy}{dx} + \frac{1}{x} = \frac{1}{x^2} \quad \text{...(i)}$$

Put $e^{-y} = z$, then $$-e^{-y} \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (i) becomes

$$-\frac{dz}{dx} + z \cdot \frac{1}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x} \cdot z = -\frac{1}{x^2}$$

which is linear in z. $$P = -\frac{1}{x},\ Q = -\frac{1}{x^2}$$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}$$

$\therefore$ The solution is $$z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x}\, dx + c$$

or $$e^{-y} \cdot \frac{1}{x} = -\int \frac{1}{x^3}\, dx + c \quad \text{or} \quad e^{-y} \cdot \frac{1}{x} = \frac{1}{2x^2} + c$$

or $$2x = e^y + 2cx^2 e^y.$$

(ii) The given equation is $\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$

Dividing throughout by $\sqrt{y}$,

$$y^{1/2} \cdot \frac{dy}{dx} + \frac{x}{1-x^2} y^{1/2} = x \quad \text{...(i)}$$

Put $y^{1/2} = z$; then $$\frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (i) becomes

$$2 \cdot \frac{dz}{dx} + \frac{x}{1-x^2} \cdot z = x \quad \text{or} \quad \frac{dz}{dx} + \frac{x}{2(1-x^2)} \cdot z = \frac{x}{2}$$

which is linear in z.
$$P = \frac{x}{2(1-x^2)},\ Q = \frac{x}{2}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{x}{2(1-x^2)}dx} = e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx} \qquad \text{| Note}$$

$$= e^{-\frac{1}{4}\log(1-x^2)} = e^{\log(1-x^2)^{-1/4}} = (1-x^2)^{-1/4}$$

$\therefore$ The solution is
$$z\,.\,(1-x^2)^{-1/4} = \int \frac{x}{2}(1-x^2)^{-1/4}\,dx + c$$

or
$$\sqrt{y}\,.\,(1-x^2)^{-1/4} = -\frac{1}{4}\int -2x(1-x^2)^{-1/4}\,dx + c$$

or
$$\sqrt{y}\,.\,(1-x^2)^{-1/4} = -\frac{1}{4}\cdot\frac{(1-x^2)^{3/4}}{\frac{3}{4}} + c$$

or
$$\sqrt{y} = -\tfrac{1}{3}(1-x^2) + c(1-x^2)^{1/4}.$$

Example 3. *Solve:* $(x^2y^3 + xy)\,dy = dx.$

Sol. The given equation is $(x^2y^3 + xy)dy = dx$

or
$$\frac{dx}{dy} = x^2y^3 + xy$$

or
$$\frac{dx}{dy} - xy = x^2y^3 \qquad \left|\text{Form } \frac{dx}{dy} + Px = Qx^n\right.$$

Dividing throughout by x^2
$$x^{-2}\frac{dx}{dy} - x^{-1}y = y^3 \qquad \ldots(i)$$

Put $x^{-1} = z$, then $\quad -x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$

$\therefore$ (*i*) becomes
$$-\frac{dz}{dy} - zy = y^3 \quad \text{or} \quad \frac{dz}{dy} + y.z = -y^3$$

which is linear in z.
$$P = y,\ Q = -y^3$$
$$\text{I.F.} = e^{\int y\,dy} = e^{y^2/2}$$

$\therefore$ The solution is
$$z\,.\,e^{1/2y^2} = \int -y^3\,.\,e^{1/2y^2}\,dy + c$$

or
$$x^{-1}\,.\,e^{1/2y^2} = -\int y^2\,.\,y\,.\,e^{1/2y^2}\,dy + c$$
$$= -\int 2t\,e^t\,dt + c, \quad \text{where } t = \tfrac{1}{2}y^2$$

or
$$x^{-1}\,.\,e^{1/2y^2} = -2e^t(t-1) + c$$

or
$$x^{-1}\,.\,e^{1/2y^2} = -2e^{1/2y^2}\left(\tfrac{1}{2}y^2 - 1\right) + c \quad \text{or} \quad x^{-1} = -y^2 + 2 + ce^{-1/2y^2}.$$

Example 4. *Solve the following:*

(*i*) $(x+1)\dfrac{dy}{dx} + 1 = 2e^{-y}$ $\qquad$ (*ii*) $\dfrac{dy}{dx} = e^{x-y}(e^x - e^y)$

Sol. (*i*) The given equation is $(x+1)\dfrac{dy}{dx} + 1 = 2e^{-y}$

or $$\frac{dy}{dx} + \frac{1}{x+1} = \frac{2e^{-y}}{x+1}$$

or $$e^y \cdot \frac{dy}{dx} + \frac{1}{x+1} \cdot e^y = \frac{2}{x+1} \qquad \ldots(i)$$

Put $e^y = z$, then $$e^y \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ From (*i*), $$\frac{dz}{dx} + \frac{1}{x+1} \cdot z = \frac{2}{x+1}$$

which is linear in z. $$P = \frac{1}{x+1}, \; Q = \frac{2}{x+1}$$

$$\text{I.F.} = e^{\int \frac{1}{x+1} dx} = e^{\log(x+1)} = x+1$$

$\therefore$ The solution is $$z(x+1) = \int \frac{2}{x+1} \cdot (x+1)\, dx + c$$

or $$e^y \cdot (x+1) = 2x + c.$$

(*ii*) The given equation is

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y) \quad \text{or} \quad \frac{dy}{dx} = e^{2x} \cdot e^{-y} - e^x$$

or $$\frac{dy}{dx} + e^x = e^{2x} \cdot e^{-y} \quad \text{or} \quad e^y \cdot \frac{dy}{dx} + e^x \cdot e^y = e^{2x} \qquad \ldots(i)$$

Put $e^y = z$, then $$e^y \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (*i*) becomes $$\frac{dz}{dx} + e^x \cdot z = e^{2x}$$

which is linear in z. $$P = e^x, \; Q = e^{2x}$$

$$\text{I.F.} = e^{\int e^x dx} = e^{e^x}$$

$\therefore$ The solution is $$z \cdot e^{e^x} = \int e^{2x} \cdot e^{e^x}\, dx + c$$

or $$e^y \cdot e^{e^x} = \int e^x \cdot e^x \cdot e^{e^x}\, dx + c$$

$$= \int t\, e^t\, dt + c, \quad \text{where } t = e^x$$

$$= e^t(t-1) + c$$

or $$e^y \cdot e^{e^x} = e^{e^x}(e^x - 1) + c \quad \text{or} \quad e^y = e^x - 1 + c\, e^{-e^x}.$$

Example 5. *Solve:* $\dfrac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0.$

Sol. The given equation is

$$\frac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$$

or $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} + 2x \tan^{-1} y - x^3 = 0$$

or $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} + 2x \tan^{-1} y = x^3 \quad \text{...(i)}$$

Put $\tan^{-1} y = z$, then $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ From (i), $$\frac{dz}{dx} + 2xz = x^3$$

which is linear in z.
$$P = 2x,\ Q = x^3$$
$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is $$z \,.\, e^{x^2} = \int x^3 \,.\, e^{x^2}\, dx + c$$

or $$\tan^{-1} y \,.\, e^{x^2} = \tfrac{1}{2} \int 2x \,.\, x^2 e^{x^2}\, dx + c$$
$$= \tfrac{1}{2} \int t e^t\, dt + c, \quad \text{where } t = x^2$$
$$= \tfrac{1}{2} e^t (t-1) + c$$

or $$\tan^{-1} y \,.\, e^{x^2} = \tfrac{1}{2} e^{x^2} (x^2 - 1) + c$$

or $$\tan^{-1} y = \tfrac{1}{2} (x^2 - 1) + ce^{-x^2}.$$

Example 6. *Solve the following differential equations:*

(i) $(x^3y^2 + xy)\, dx = dy$ *(ii)* $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$

Sol. *(i)* The given equation is $(x^3y^2 + xy)\, dx = dy$

or $$\frac{dy}{dx} = x^3y^2 + xy \quad \text{or} \quad \frac{dy}{dx} - xy = x^3y^2$$

Dividing both sides by y^2, $$y^{-2} \frac{dy}{dx} - xy^{-1} = x^3 \quad \text{...(i)}$$

Put $y^{-1} = z$, then $$-y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (i) becomes $$\frac{dz}{dx} - xz = x^3 \quad \text{or} \quad \frac{dz}{dx} + xz = -x^3$$

which is linear in z.
$$P = x, \quad Q = -x^3$$
$$\text{I.F.} = e^{\int x\,dx} = e^{\frac{x^2}{2}}$$

$\therefore$ The solution is $$z \,.\, e^{\frac{x^2}{2}} = \int -x^3 \,.\, e^{\frac{x^2}{2}}\, dx + c = -\int x^2 \,.\, xe^{\frac{x^2}{2}}\, dx + c$$
$$= -\int 2te^t\, dt + c, \quad \text{where } t = \frac{x^2}{2}$$
$$= -\int 2te^t dt + c = -2e^t (t-1) + c$$
$$y^{-1} \,.\, e^{\frac{x^2}{2}} = -2e^{\frac{x^2}{2}} \left(\frac{x^2}{2} - 1 \right) + c$$

or $$y^{-1} = -x^2 + 2 + ce^{-\frac{x^2}{2}}.$$

(ii) The given equation is $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$

Dividing both sides by $y(\log y)^2$, we get

$$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} + \frac{1}{\log y} \cdot \frac{1}{x} = \frac{1}{x^2} \quad \ldots(i)$$

Put $\frac{1}{\log y} = (\log y)^{-1} = z$, then $-(\log y)^{-2} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$

or
$$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} = -\frac{dz}{dx}$$

$\therefore$ From (i), $\quad -\frac{dz}{dx} + z \cdot \frac{1}{x} = \frac{1}{x^2}$ or $\frac{dz}{dx} - \frac{1}{x} z = -\frac{1}{x^2}$

which is linear in z. $\quad P = -\frac{1}{x}, Q = -\frac{1}{x^2}$.

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is $\quad z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x} dx + c$

or
$$z \cdot \frac{1}{x} = -\int x^{-3} dx + c = -\frac{x^{-2}}{-2} + c$$

or
$$\frac{1}{\log y} \cdot \frac{1}{x} = \frac{1}{2x^2} + c \quad \text{or} \quad \frac{1}{\log y} = \frac{1}{2x} + cx.$$

Example 7. *Show how to solve an equation of the form*

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \quad \textit{where P, Q are functions of x only.}$$

Sol. (a) The given equation is

$$f'(y)\frac{dy}{dx} + Pf(y) = Q \quad \ldots(i)$$

where P, Q are functions of x only.

Put $f(y) = z$, then $\quad f'(y)\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (i) becomes $\quad \frac{dz}{dx} + Pz = Q$

which is linear in z and can be solved.

I.F. $= e^{\int P dx}$ and the solution is

$$z(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$$

or
$$f(y) \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$$

Example 8. *Solve the following differential equations:*

(i) $(x+1)\frac{dy}{dx}+1=e^{x-y}$ (ii) $\frac{dy}{dx}=y\tan x-y^2\sec x.$

Sol. (i) The given equation is

$$(x+1)\frac{dy}{dx}+1=\frac{e^x}{e^y} \quad \text{or} \quad e^y\frac{dy}{dx}+\frac{e^y}{x+1}=\frac{e^x}{x+1} \qquad ...(i)$$

Putting $e^y=z$ so that $e^y\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ (i) becomes $\frac{dz}{dx}+\frac{z}{x+1}=\frac{e^x}{x+1}$

which is linear in z with $P=\frac{1}{x+1}, Q=\frac{e^x}{x+1}$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int\frac{1}{x+1}dx}=e^{\log(x+1)}=x+1$$

$\therefore$ The solution is $z(x+1)=\int\frac{e^x}{x+1}\cdot(x+1)\,dx+c$ or $e^y(x+1)=e^x+c.$

(ii) The given equation is

$$\frac{dy}{dx}-y\tan x=-y^2\sec x$$

or $$-\frac{1}{y^2}\cdot\frac{dy}{dx}+\frac{1}{y}\tan x=\sec x \qquad ...(1)$$

Putting $\frac{1}{y}=z$ so that $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ Equation (1) becomes $\frac{dz}{dx}+z\tan x=\sec x$

which is linear in z with $P=\tan x,\quad Q=\sec x$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int\tan x\,dx}=e^{\log\sec x}=\sec x$$

$\therefore$ The solution is $z\cdot\sec x=\int\sec x\cdot\sec x\,dx+c$

or $\frac{1}{y}\sec x=\tan x+c$ or $\frac{1}{y}=\sin x+c\cos x.$

TEST YOUR KNOWLEDGE

Solve the following differential equations:

1. $\frac{dy}{dx}+\frac{y}{x}=y^2$
2. $y'+y=y^2$
3. $\frac{dy}{dx}=x^3y^3-xy$
4. $3\frac{dy}{dx}+\frac{2}{x+1}y=\frac{x^3}{y^2}$

5. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^6$

6. $x\dfrac{dy}{dx} + y = x^3y^4$

7. $\dfrac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

8. $(y \log x - 1)y\, dx = x\, dy$

9. $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

10. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\, e^x \sec y$

11. $(x - y^2)\, dx + 2xy\, dy = 0$

12. $\cos x\, dy = y(\sin x - y)\, dx$

13. $xy - \dfrac{dy}{dx} = y^3e^{-x^2}$

14. $(xy^2 - e^{1/x^3})\, dx - x^2y\, dy = 0$

15. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$

16. $(xy - 2x \log x)\, dy = 2y\, dx$

17. $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 \log x$

18. $y\,(2xy + e^x)\, dx = e^x y$

19. $x\dfrac{dy}{dx} + y = y^2x^3 \cos x$

20. $\sin y \dfrac{dy}{dx} = \cos y\,(1 - x \cos y)$

Answers

1. $\dfrac{1}{xy} + \log x = c$

2. $y = \dfrac{1}{1 + ce^x}$

3. $y^{-2} = x^2 + 1 + ce^{x^2}$

4. $y^2(x + 1)^2 = \dfrac{x^6}{6} + \dfrac{2x^5}{5} + \dfrac{x^4}{4} + c$

5. $\dfrac{1}{y^5} = \dfrac{5}{2}x^3 + cx^5$

6. $\dfrac{1}{y^3} = -3x^3 \log x + cx^3$

7. $\dfrac{1}{y}\sec^2 x = -\dfrac{\tan^3 x}{3} + c$

8. $\dfrac{1}{y} = \log x + 1 + cx$

9. $\tan y = \dfrac{1}{2}(x^2 - 1) + ce^{-x^2}$

10. $\sin y = (1 + x)(e^x + c)$

11. $y^2 = x\,(c - \log x)$

12. $\dfrac{1}{y} = \sin x + c \cos x$

13. $y^{-2} \cdot e^{x^2} = 2x + c$

14. $3y^2 = 2x^2e^{\frac{1}{x^3}} + cx^2$

15. $e^{x+y} = \dfrac{1}{2}e^{2x} + c$

16. $y \log x = \dfrac{y^2}{4} + c$

17. $\dfrac{1}{y} = -\dfrac{1}{2}(\log x)^2 + cx$

18. $e^x = y(c - x^2)$

19. $\dfrac{1}{xy} = -x \sin x - \cos x + c$

20. $\sec y = x + 1 + ce^x$

□□□

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0.$

Sol. The given equation is $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

Factorising, $(xp + 3y)(xp - 2y) = 0$

$\Rightarrow$ $xp + 3y = 0$ or $xp - 2y = 0$

Now, $xp + 3y = 0$

$\Rightarrow$ $x\frac{dy}{dx} + 3y = 0$ or $\frac{dy}{y} + 3\frac{dx}{x} = 0$

Integrating, $\log y + 3\log x = \log c$ or $x^3y = c$

Also, $xp - 2y = 0$

$\Rightarrow$ $x\frac{dy}{dx} - 2y = 0$ or $\frac{dy}{y} - 2\frac{dx}{x} = 0$

Integrating, $\log y - 2\log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

$\therefore$ The general solution of the given equation is $(x^3y - c)(y - cx^2) = 0$.

Example 2. *Solve* $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0.$

Sol. Solving the given equation for p, we have

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$= \frac{(2y^2 - 3x^2) \pm (3x^2 + 2y^2)}{2xy} = \frac{2y}{x} \text{ or } -\frac{3x}{y}$$

Now, $p = \frac{2y}{x} \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$ or $\frac{dy}{y} - \frac{2dx}{x} = 0$

Integrating, $\log y - 2\log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

Also, $p = -\frac{3x}{y} \Rightarrow \frac{dy}{dx} = -\frac{3x}{y}$ or $ydy + 3xdx = 0$

Integrating, $\frac{y^2}{2} + \frac{3x^2}{2} = C$ or $y^2 + 3x^2 = c$

$\therefore$ The general solution of the given equation is $(y - cx^2)(y^2 + 3x^2 - c) = 0$.

Example 3. *Solve* $p^2 + 2py\cot x = y^2$.

Sol. The given equation can be written as $(p + y\cot x)^2 = y^2(1 + \cot^2 x)$

$p + y\cot x = \pm y\operatorname{cosec} x$

$\therefore$ The component equations are

$$p = y(-\cot x + \operatorname{cosec} x) \quad ...(1)$$

$$p = y(-\cot x - \operatorname{cosec} x) \quad ...(2)$$

From (1), $$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

or $$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\, dx$$

Integrating, $$\log y = -\log \sin x + \log \tan \frac{x}{2} + \log c = \log \frac{c \tan \frac{x}{2}}{\sin x}$$

or $$y = \frac{c \tan \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{c}{2 \cos^2 \frac{x}{2}} = \frac{c}{1 + \cos x}$$

or $$y(1 + \cos x) = c.$$

From (2), $$\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

or $$\frac{dy}{y} = (-\cot x - \operatorname{cosec} x)\, dx$$

Integrating, $$\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$$

or $$y = \frac{c}{2 \sin^2 \frac{x}{2}} = \frac{c}{1 - \cos x} \quad \text{or} \quad y(1 - \cos x) = c$$

$\therefore$ The general solution of the given equation is
$$[y(1 + \cos x) - c]\,[y(1 - \cos x) - c] = 0.$$

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $p^2 - 7p + 12 = 0$
2. $xy \left(\frac{dy}{dx}\right)^2 - (x^2 + y^2) \frac{dy}{dx} + xy = 0$
3. $yp^2 + (x - y)p - x = 0$
4. $x^2 \left(\frac{dy}{dx}\right)^2 + 3xy \frac{dy}{dx} + 2y^2 = 0$
5. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$
6. $p^2 - 2p \sinh x - 1 = 0$
7. $p(p + y) = x(x + y)$
8. $4y^2p^2 + 2pxy(3x + 1) + 3x^3 = 0.$

Answers

1. $(y - 4x - c)(y - 3x - c) = c$
2. $(y^2 - x^2 - c)(y - cx) = 0$
3. $(y - x - c)(x^2 + y^2 - c) = 0$
4. $(xy - c)(x^2y - c) = 0$
5. $(xy - c)(x^2 - y^2 - c) = 0$
6. $(y - e^x - c)(y - e^{-x} - c) = 0$
7. $(y - \frac{1}{2}x^2 + c)(y + x + ce^{-x} - 1) = 0$
8. $(y^2 + x^3 - c)(y^2 + \frac{1}{2}x^2 - c) = 0.$

3.3 EQUATIONS SOLVABLE FOR y

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus the equations of this type can be put as $y = f(x, p)$ ...(1)

Differentiating (1) w.r.t. x, we get $\dfrac{dy}{dx} = p = F\left(x, p, \dfrac{dp}{dx}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and x.

Suppose the solution of (2) is $\phi(x, p, c) = 0$ ...(3)

Now elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1(p, c),\; y = \phi_2(p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y + px = x^4p^2$.

Sol. Given equation is $y = -px + x^4p^2$ ...(1)

Differentiating both sides w.r.t. x,

$$\frac{dy}{dx} = p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

or $$2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

or $$\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$$

Discarding the factor $(1 - 2px^3)$, which does not involve $\dfrac{dp}{dx}$, we have

$$2p + x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{p} + 2\frac{dx}{x} = 0$$

Integrating, $\log p + 2\log x = \log c$

or $$\log px^2 = \log c$$

or $$px^2 = c \Rightarrow p = \frac{c}{x^2}.$$

Putting this value of p in (1), we have $y = -\dfrac{c}{x} + c^2$.

Example 2. *Solve* $y = 2px - p^2$.

Sol. The given equation is $y = 2px - p^2$ ...(1)

Differentiating both sides w.r.t. x, $\dfrac{dy}{dx} = p = 2p + 2x\dfrac{dp}{dx} - 2p\dfrac{dp}{dx}$

or $$p + (2x - 2p)\frac{dp}{dx} = 0$$

or $$p\frac{dx}{dp} + 2x - 2p = 0$$

or
$$\frac{dx}{dp} + \frac{2}{p}x = 2 \qquad ...(2)$$
which is a linear equation.
$$\text{I.F.} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$
∴ The solution of (2) is $x\,\text{I.F.} = \int 2\,\text{I.F.}\,dp + c$

or
$$xp^2 = \int 2p^2 dp + c$$
or
$$xp^2 = \frac{2}{3}p^3 + c \quad \text{or} \quad x = \frac{2}{3}p + cp^{-2} \qquad ...(3)$$
Putting this value of x in (1), we have $y = 2p\left(\frac{2}{3}p + cp^{-2}\right) - p^2$

or
$$y = \frac{1}{3}p^2 + 2cp^{-1}. \qquad ...(4)$$
Equations (3) and (4) together constitute the general solution of (1).

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $xp^2 - 2yp + ax = 0$
2. $y - 2px = \tan^{-1}(xp^2)$
3. $16x^2 + 2p^2y - p^3x = 0$
4. $y = x + 2\tan^{-1} p$
5. $y = 3x + \log p$
6. $x - yp = ap^2$
7. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0.$
8. $y = 2px - xp^2$

Answers

1. $2y = cx^2 + \frac{a}{c}$
2. $y = 2\sqrt{cx} + \tan^{-1} c$
3. $16 + 2c^2y - c^3x^2 = 0$
4. $x = \log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1} p + c,\; y = \log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1} p + c$
5. $y = 3x + \log\frac{3}{1 - ce^{3x}}$
6. $x = \frac{p}{\sqrt{1-p^2}}(c + a\sin^{-1} p),\; y = \frac{1}{\sqrt{1-p^2}}(c + a\sin^{-1} p) - ap.$
7. $y = c^2 + 2\sqrt{cx}.$
8. $y = 2\sqrt{cx} - c.$

3.4 EQUATIONS SOLVABLE FOR *x*

If the equation is solvable for x, we can express x explicitly in terms of y and p. Thus, the equations of this type can be put as $x = f(y, p)$ ...(1)

Differentiating (1) w.r.t. y, we get $\frac{dx}{dy} = \frac{1}{p} = F\left(y, p, \frac{dp}{dy}\right)$ ...(2)

Equation (2) is a differential equation of first order in p and y.

Suppose the solution of (2) is $\phi(y, p, c) = 0$ ...(3)

(5)... Now elimination of p from (1) and (3) gives the required solution.

If p cannot be easily eliminated, then we solve equations (1) and (3) for x and y to get

$$x = \phi_1 (p, c), \ y = \phi_2 (p, c)$$

These two relations together constitute the solution of the given equation with p as parameter.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $y = 2px + y^2p^3$.

Sol. Solving for x, we have $x = \dfrac{1}{2}\left(\dfrac{y}{p} - y^2p^2\right)$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\cdot\frac{dp}{dy} - 2yp^2 - 2y^2p\frac{dp}{dy}\right)$$

or $$2p = p - y\frac{dp}{dy} - 2yp^4 - 2y^2p^3\frac{dp}{dy}$$

or $$p + 2yp^4 + y\frac{dp}{dy} + 2y^2p^3\frac{dp}{dy} = 0 \quad \text{or} \quad p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0$$

or $$\left(p + y\frac{dp}{dy}\right)(1 + 2yp^3) = 0$$

Discarding the factor $(1 + 2yp^3)$ which does not involve $\dfrac{dp}{dy}$, we have

$$p + y\frac{dp}{dy} = 0 \quad \text{or} \quad \frac{dy}{y} + \frac{dp}{p} = 0$$

Integrating, $\log y + \log p = \log c$ or, $py = c$ or $p = \dfrac{c}{y}$

Putting this value of p in the given equation, we have

$$y = \frac{2cx}{y} + \frac{c^3}{y} \quad \text{or} \quad y^2 = 2cx + c^3$$

which is the required solution.

Example 2. *Solve* $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$.

Sol. Solving for x, we have $x = \tan^{-1} p + \dfrac{p}{1+p^2}$ ...(1)

Differentiating both sides w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{1+p^2}\cdot\frac{dp}{dy} + \frac{(1+p^2) - 2p^2}{(1+p^2)^2}\cdot\frac{dp}{dy}$$

or $$\frac{1}{p} = \frac{2(1+p^2)-2p^2}{(1+p^2)^2}\frac{dp}{dy} \quad \text{or} \quad dy = \frac{2p}{(1+p^2)^2}dp$$

Integrating, $$y = c - \frac{1}{1+p^2} \qquad \text{...(2)}$$

Equations (1) and (2) together constitute the general solution.

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $y = 3px + 6p^2y^2$
2. $y = 2px + p^2y$
3. $p^3 - 4xyp + 8y^2 = 0$
4. $y^2 \log y = xyp + p^2$
5. $x = y + a \log p$
6. $x = y + p^2$

Answers

1. $y^3 = cx + 6c^2$
2. $y^2 = 2cx + c^2$
3. $64y = c(c-4x)^2$
4. $\log y = cx + c^2$
5. $x = c + a \log \frac{p}{p-1}$, $y = c - a \log (p-1)$
6. $x = -p - 2 \log (1-p) + c$, $y = -p^2 - 2p - 2 \log (1-p) + c$.

3.5 CLAIRAUT'S EQUATION

An equation of the form $y = px + f(p)$ ...(1)
is known as Clairaut's equation.

Differentiating (1) w.r.t. x, we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \quad \text{or} \quad [x + f'(p)]\frac{dp}{dx} = 0$$

Discarding the factor $[x + f'(p)]$, we have $\frac{dp}{dx} = 0$

Integrating, $p = c$

Putting $p = c$ in (1), the required solution is $y = cx + f(c)$

Thus the solution of Clairaut's equation is obtained by writing c for p.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $(y - px)(p - 1) = p$.

Sol. The given equation can be written as

$$y - px = \frac{p}{p-1} \quad \text{or} \quad y = px + \frac{p}{p-1}$$

This is of Clairaut's form. Hence putting c for p, the solution is $y = cx + \frac{c}{c-1}$.

Many differential equations can be reduced to Clairaut's form by suitably changing the variables