University Question Papers solutions

(2018, 2019, 2022) Mathematics - I (BSC-103-E)

Section (3, 4, 5)

Vector space

Short Answer Type questions

Q No 1 Define inner product space

Solu. Let V be a vector space over a field F. Let $a, b \in F$ and $u, v, w \in V$ be arbitrary. The vector space V is called an inner product space if there exist a function $\langle , \rangle : V \times V \to F$ satisfying the following

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$

(ii) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

(iii) $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$

The function $\langle , \rangle$ satisfying (i), (ii), (iii) is called an inner product on V.

Thus vector space V with an inner product is called an inner product space.

Q.No.2 State Rank - Nullity Theorem

Solu. Let U, V be a vector space where U is finite dimensional. Let $T : U \to V$ be a Linear transformation. Then

$$\boxed{\text{Rank}(T) + \text{Nullity}(T) = \dim U}$$

Q.No.3 Show that a fun. $T : R^2 \to R^3$ given by $T(x_1, x_2) = (x_1 - x_2, 2x_2 - x_1, -x_1)$ is a Linear Transformation.

Soln $T(x_1, x_2) = (x_1 - x_2, 2x_2 - x_1, -x_1)$

let $u_1 = (x_1, x_2)$, $u_2 = (y_1, y_2) \in U$

$a \in F$

Now (i) $T(u_1 + u_2) = T((x_1, x_2) + (y_1, y_2))$

$= T(x_1 + y_1, x_2 + y_2)$

$= (x_1 + y_1 - x_2 - y_2, 2x_2 + 2y_2 - x_1 - y_1, -x_1 - y_1)$

$= (x_1 - x_2, 2x_2 - x_1, -x_1) + (y_1 - y_2, 2y_2 - y_1, -y_1)$

$= T(x_1, x_2) + T(y_1, y_2)$

$\boxed{T(u_1 + u_2) = T(u_1) + T(u_2)}$

(ii) $T(au_1) = T(a(x_1, x_2)) = T(ax_1, ax_2)$

$= (ax_1 - ax_2, 2ax_2 - ax_1, -ax_1)$

$= (a(x_1 - x_2), a(2x_2 - x_1), -ax_1)$

$= a(x_1 - x_2, 2x_2 - x_1, -x_1)$

$= aT(x_1, x_2)$

$\boxed{T(au_1) = aT(u_1)}$

Both conditions are true so given transformation is a linear Transformation.

Q.No.4 Check the vectors $(2,1,1), (2,0,-1), (4,2,1)$ L.D. or not?

Soln $a(2,1,1) + b(2,0,-1) + c(4,2,1) = 0$

$2a + 2b + 4c = 0$ - (1)

$a + 2c = 0$ - (2)

$a + b + c = 0$ - (3)

Scanned with CamScanner

eq. 1 + 2 × eq. (3)

$$2a + \cancel{b} + 4c = 0$$
$$2a \cancel{+ 2b} + 2c = 0$$
$$4a + 6c = 0 \quad -(4)$$

eq. (4) − 3 eq. (2)

$$4a + \cancel{6c} = 0$$
$$3a + \cancel{6c} = 0$$
$$\boxed{a = 0}$$

Put $a = 0$ in eq. (2) $\quad 0 + 2c = 0 \Rightarrow \boxed{c = 0}$

Put these values in (1)

$$2 \times 0 + 2b + 4 \times 0 = 0$$
$$\boxed{b = 0}$$

Hence $a = b = c = 0$

So vector's are L.I. Answer

Q.No.5. Let $V = R^2 = \{(a_1, a_2) : a_1, a_2 \in R\}$ Test whether $W = \{(x, y) \mid 2x + 3y = 2 : x, y \in R\}$ is its subspace or not?

Solu. First check $(0,0) \in W$?

$$2x + 3y = 2$$

at $(0,0)$ $\quad 2 \times 0 + 3 \times 0 = 2$

$$0 \neq 2$$

So $(0,0) \notin W$

So W is not subspace Answer

Q.No.6. In a vector space V over F, Prove that $au = bu$ that implies $a = b$ if $u \neq 0$, $a, b \in F$

Solu.

$$au = bu$$
$$au - bu = 0$$
$$(a - b)u = 0$$

$\Rightarrow \quad u \neq 0$

$\Rightarrow \quad a-b=0$

$\Rightarrow a=b$ Answer

Q.No 7 Check whether the vectors $(1,0,-1), (2,5,1), (0,-4,3)$ in R^3 are L.I. or not

Soln. Here
$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & -4 & 3 \end{vmatrix} = 1(15+4)-1(-8) = 19+8 = 27 \neq 0$$

Hence vectors are $\underline{L.I.}$ Answer

Q.No.8 If $T: R^4 \to R^2$ is a L.T. such that Nullity of $T = 1$. Find Rank of T.

Soln By Rank - Nullity Theorem

$\rho(T) + \mu(T) = \dim R^4 = 4$

But $\mu(T) = 1$ given,

So $\rho(T) + 1 = 4$

$\rho(T) = 3$

So $\boxed{\text{Rank of } T \; (\rho(T)) = 3}$ Answer

Big Answer Type Questions

Q.No1 Apply the Gram - Schmidt process to the vector $u_1 = (1,0,1)$, $u_2 = (1,0,-1)$, $u_3 = (0,3,4)$ to obtain an orthonormal basis for $R^3(R)$ with the standard inner product.

Soln. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3) \in R^3$

So standart inner product

$$\boxed{\langle u, v \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3}$$

Here

$u_1 = (1,0,1)$, $u_2 = (1,0,-1)$, $u_3 = (0,3,4)$

now $v_1 = u_1 = (1,0,1)$

$\|v_1\|^2 = \langle v_1, v_1 \rangle = 1^2 + 0^2 + 1^2 = 2$

$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle v_1}{\|v_1\|^2}$

$= u_2 - \frac{\langle u_2, u_1 \rangle u_1}{2}$ | $\langle u_2, u_1 \rangle = 1\times1 + 0\times0 + 1(-1) = 0$

$v_2 = u_2 - 0 = u_2 = (1,0,-1)$

$\|v_2\|^2 = \langle v_2, v_2 \rangle = 1^2 + 0^2 + (-1)^2 = 2$

$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}$

$= u_3 - \frac{\langle u_3, u_1 \rangle u_1}{2} - \frac{\langle u_3, u_2 \rangle u_2}{2}$ | $\langle u_3, u_1 \rangle = 1\times0 + 3\times0 + 4\times1 = 4$; $\langle u_3, u_2 \rangle = 0\times1 + 3\times0 + 4(-1) = -4$

$v_3 = u_3 - \frac{\cancel{4}^{2} u_1}{\cancel{2}} - \frac{(-\cancel{4})^{2} u_2}{\cancel{2}}$

$v_3 = u_3 - 2u_1 + 2u_2$

$= (0,3,4) - 2(1,0,1) + 2(1,0,-1)$

$= (0,3,4) + (-2,0,-2) + (2,0,-2)$

$v_3 = (0,3,0)$

$\|v_3\|^2 = 0^2 + 3^2 + 0^2 = 9$

$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$w_2 = \frac{v_2}{\|v_2\|} = \frac{(1,0,-1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

$w_3 = \frac{v_3}{\|v_3\|} = \frac{(0,3,0)}{3} = (0,1,0)$

Hence Required orthonormal Basis $= \{w_1, w_2, w_3\} = \left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), (0,1,0)\right\}$

Qn 2 Let $T: R^3 \to R^2$ be defined by $T(x,y,z) = (x-y, 2z)$
$\forall (x,y,z) \in R^3$
Test whether T is a Linear. If so, Find ~~rank~~ rank and Nullity of T.

Soln
$T: R^3 \to R^2$
$T(x,y,z) = (x-y, 2z)$

Let $u_1 = (x_1, y_1, z_1), u_2 = (x_2, y_2, z_2) \in R^3$
$a, b \in F$.

Now (i) $T(u_1+u_2) = T((x_1,y_1,z_1)+(x_2,y_2,z_2))$
$= T(x_1+x_2, y_1+y_2, z_1+z_2)$
$= (x_1+x_2-y_1-y_2, 2(z_1+z_2))$
$= (x_1-y_1+x_2-y_2, 2z_1+2z_2)$
$= (x_1-y_1, 2z_1) + (x_2-y_2, 2z_2)$
$= T(x_1,y_1,z_1) + T(x_2,y_2,z_2)$
$= T(u_1) + T(u_2)$

$\Rightarrow T(u_1+u_2) = T(u_1)+T(u_2)$

(ii) $T(a \cdot u_1) = T(a(x_1,y_1,z_1))$
$= T(ax_1, ay_1, az_1)$
$= (ax_1 - ay_1, az_1)$
$= (a(x_1-y_1), az_1)$
$= a((x_1-y_1), z_1)$
$= a\,T(x_1,y_1,z_1)$
$= a\,T(u_1)$

$\Rightarrow \boxed{T(au_1) = a\,T(u_1)}$

Hence T is Linear
Now find Rank and Nullity.

First find N(T) -

let $u = (x, y, z) \in R^3$

Now $T(u) = 0$

$T(x, y, z) = 0$

$(x-y, 2z) = (0, 0)$

$x - y = 0, \; 2z = 0$

$x = y, \; z = 0$

so $u = (x, y, z) = (x, x, 0) = x(1, 1, 0)$

hence $u = (1, 1, 0)$

$N(T) = \{ (1, 1, 0) \}$

$\boxed{\dim N(T) = \mathcal{N}(T) = 1}$

Find R(T)

let $u = (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$

$v = T(u) = T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1)$

But $v = T(u) \in R(T)$

so $R(T) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1)$

$= x(1, 0) + y(-1, 0) + z(0, 2)$

$T(1, 0, 0) = (1, 0)$
$T(0, 1, 0) = (-1, 0)$
$T(0, 0, 1) = (0, 2)$

$\Rightarrow R(T)$ is spanned by vectors $(1, 0), (-1, 0)$ & $(0, 2)$

now $\dim R^3 = 3$.

Now check for L.I.

First check $(1, 0), (-1, 0) \Rightarrow \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = 0$ so L.D.

so we check $(1, 0), (0, 2) \Rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \neq 0$ so L.I.

Hence R(T) is the basis of $\{(1, 0), (0, 2)\}$

$\boxed{\rho(T) = \dim R(T) = 2}$

Hence $\rho(T) + \mathcal{N}(T) = \dim R^3 = 3$

$2 + 1 = 3$

Q. No. 3 Find the basis and dim. of the subspace spanned by $(1,0,-1)$, $(2,5,1)$, $(0,-4,3)$ in R^3

Soln. First check $(1,0,-1)$ & $(2,5,1)$ as L.I.

$$a(1,0,-1) + b(2,5,1) = 0$$

$$\left.\begin{array}{l} a+2b=0 \\ 5b=0 \\ -a+b=0 \end{array}\right] \Rightarrow b=0 \Rightarrow a=0$$

Hence L.I.

Now check $(1,0,-1)$ & $(2,5,1)$ & $(0,-4,3)$

$$a(1,0,-1) + b(2,5,1) + c(0,-4,3) = 0$$

$$a+2b=0 \quad -(1)$$

$$5b-4c=0 \quad -(2)$$

$$-a+b+3c=0 \quad -(3)$$

eq (3) + eq (1)

$$\begin{array}{l} -\cancel{a}+b+3c=0 \\ \cancel{a}+2b \quad\;\; =0 \\ \hline 3b+3c=0 \end{array}$$

$$b+c=0 \quad -(4)$$

eq. (2) + 4·eq (4)

$$\begin{array}{l} 5b-\cancel{4c}=0 \\ 4b+\cancel{4c}=0 \\ \hline 9b=0 \end{array} \Rightarrow b=0 \text{ Put in } (4) \quad c=0$$

Put $b=c=0$ in (3) $\Rightarrow a=0$

$\Rightarrow a=b=c=0$

So $W=\{(1,0,-1),(2,5,1),(0,-4,3)\}$

$\dim(W) = 3$ Ans

Q.No 4 Find an orthonormal basis of the inner product space $R^3(R)$ with standard inner product, given the basis $B = \{(1,1,0), (1,-1,1), (-1,1,2)\}$ using Gram-Schmidt orthogonalization process. Also find Fourier Coefficient of the vector $(2,1,3)$ relative to ~~orthonormal~~ orthonormal Basis.

Solu. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3) \in R^3$

So standard inner product

$$\langle u, v \rangle = (x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3)$$

Let $u_1 = (1,1,0)$, $u_2 = (1,-1,1)$, $u_3 = (-1,1,2)$

$$\boxed{v_1 = u_1 = (1,1,0)}$$

$$\|v_1\|^2 = \langle v_1, v_1 \rangle = 1^2 + 1^2 + 0^2 = 2$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle v_1}{\|v_1\|^2} = u_2 - \frac{\langle u_2, u_1 \rangle u_1}{2}$$

$$v_2 = u_2 - \frac{\langle u_2, u_1 \rangle u_1}{2}$$

$\langle u_2, u_1 \rangle = 1 \times 1 + (-1)(1) + 1 \times 0 = 1 - 1 + 0 = 0$

$$\Rightarrow \boxed{v_2 = u_2 = (1,-1,1)}$$

$$\|v_2\|^2 = \langle v_2, v_2 \rangle = 1 + 1 + 1 = 3$$

$$v_3 = u_3 - \frac{\langle u_3, v_1 \rangle v_1}{\|v_1\|^2} - \frac{\langle u_3, v_2 \rangle v_2}{\|v_2\|^2}$$

$$v_3 = u_3 - \frac{\langle u_3, u_1 \rangle u_1}{2} - \frac{\langle u_3, u_2 \rangle u_2}{3}$$

$\langle u_3, u_1 \rangle = -1 + 1 + 0 = 0$

$\langle u_3, u_2 \rangle = -1 - 1 + 2 = 0$

$$\boxed{v_3 = u_3 - 0 - 0 = u_3 = (-1,1,2)}$$

$$\|v_3\|^2 = (-1)^2 + 1^2 + 2^2 = 1 + 1 + 4 = 6$$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,0)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(1,-1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{(-1, 1, 2)}{\sqrt{6}} = \left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

Required orthonormal Basis $= \{w_1, w_2, w_3\}$

$$= \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\right\}$$

Now find Fourier coefficient Relative to orthonormal Basis for $(2, 1, 3)$

$$(2, 1, 3) = a\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + b\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + c\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$$

$$2 = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{3}} - \frac{c}{\sqrt{6}} \quad -(1)$$

$$1 = \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{3}} + \frac{c}{\sqrt{6}} \quad -(2)$$

$$3 = \frac{b}{\sqrt{3}} + \frac{2c}{\sqrt{6}} \quad -(3)$$

eq (1) - eq (2)

$$2 = \cancel{\frac{a}{\sqrt{2}}} + \frac{b}{\sqrt{3}} - \frac{c}{\sqrt{6}}$$

$$\underline{\overset{-}{1} = \overset{-}{\cancel{\frac{a}{\sqrt{2}}}} \overset{+}{-} \frac{b}{\sqrt{3}} \overset{-}{+} \frac{c}{\sqrt{6}}}$$

$$1 = \frac{2b}{\sqrt{3}} - \frac{2c}{\sqrt{6}} \quad -(4)$$

eq (3) & eq (4)

$$3 = \frac{b}{\sqrt{3}} + \cancel{\frac{2c}{\sqrt{6}}}$$

$$\underline{1 = \frac{2b}{\sqrt{3}} - \cancel{\frac{2c}{\sqrt{6}}}}$$

$$4 = \frac{3b}{\sqrt{3}} = \frac{3b}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}b\sqrt{3}}{\cancel{3}}$$

$$\Rightarrow \boxed{b = \frac{4}{\sqrt{3}}}$$ Put the value of b in eq. (4)

Scanned with CamScanner

$$1 = \frac{2}{\sqrt{3}} \times \frac{4}{\sqrt{3}} - \frac{2c}{\sqrt{6}}$$

$$1 = \frac{8}{3} - \frac{2c}{\sqrt{6}}$$

$$\frac{2c}{\sqrt{6}} = \frac{8}{3} - 1 = \frac{8-3}{3} = \frac{5}{3}$$

$$c = \frac{5}{3} \times \frac{\sqrt{6}}{2} = \frac{5\cancel{\sqrt{6}}}{\cancel{6}} \cancel{\times \frac{\sqrt{6}}{\sqrt{6}}} = \frac{5}{\sqrt{6}}$$

$$\boxed{c = \frac{5}{\sqrt{6}}}$$

Put the value of $b + c$ in eqn. 1

$$2 = \frac{a}{\sqrt{2}} + \frac{4}{\sqrt{3}\cdot\sqrt{3}} - \frac{5}{\sqrt{6}\cdot\sqrt{6}}$$

$$2 = \frac{a}{\sqrt{2}} + \frac{4}{3} - \frac{5}{6}$$

$$\frac{a}{\sqrt{2}} = \frac{2}{1} - \frac{4}{3} + \frac{5}{6} = \frac{12 - 8 + 5}{6} = \frac{\cancel{9}}{\cancel{6}}{}^{3}_{2} = \frac{3}{2}$$

$$a = \frac{3\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\boxed{a = \frac{3}{\sqrt{2}}}$$

Hence

$$\boxed{(2,1,3) = \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) + \frac{4}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + \frac{5}{\sqrt{6}}\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)}$$

Ans

Q.No.5 For a L.T. $T : R^2(R) \to R^3(R)$

defined by $T(x,y) = (x+y, x-y, y)$

find the basis and dim. of

(i) its range space

(ii) its null space. Also verify Rank-nullity theorem

Solu

$T: R^2 \to R^3$

$T(x, y) = (x+y, x-y, y)$

(i) Find $N(T)$

let $u = (x, y) \in R^2$

$T(u) = 0$

$T(x, y) = 0$

$(x+y, x-y, y) = (0, 0, 0)$

$\Rightarrow x+y = 0, \quad x-y = 0, \quad y = 0$

$\Rightarrow \quad x = y = 0$

hence $u = (x, y) = (0, 0)$

so $N(T) = \{(0, 0)\}$ It is the basis itself.

so $\boxed{\dim(N(T)) = 0}$

(ii) Find $R(T)$.

let $u = (x, y) = x(1, 0) + y(0, 1)$

Taking T on both sides

$v = T(u) = T(x, y) = x\,T(1, 0) + y\,T(0, 1)$ | $T(1,0) = (1, 1, 0)$, $T(0,1) = (1, -1, 0)$

$v = T(u) = x(1, 1, 0) + y(1, -1, 0)$

But $v = T(u) \in RT$

so $\boxed{R(T) = x(1, 1, 0) + y(1, -1, 0)}$

$\Rightarrow R(T)$ is spanned by vectors $(1, 1, 0), (1, -1, 0)$

now check for L.I.

$a(1, 1, 0) + b(1, -1, 0) = 0$

$a+b = 0, \ a-b = 0 \Rightarrow a = b = 0$ so L.I.

Hence $R(T) = \{ (1,1,0), (1,-1,0) \}$

$$\boxed{\dim R(T) = 2}$$

By Rank nullity theorem

$$\dim(N(T)) + \dim(R(T)) = \dim R^2 = 2$$

$$0 + 2 = 2$$
$$2 = 2$$

Ans

Q.No 6 Find the L.T $T: R^3(R) \to R^3(R)$ determined by the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ in the standard Basis also evaluate $T(-2,2,3)$ & $T(1,4,-2)$.

Solu. Let

$$B = \{(1,0,0), (0,1,0), (0,0,1)\} \in R^3 \in U$$
$$B' = \{(1,0,0), (0,1,0), (0,0,1)\} \in R^3 \in V$$

matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

So $T(1,0,0) = 1(1,0,0) + 0(0,1,0) + (-1)(0,0,1) = (1,0,-1)$

$T(0,1,0) = 2(1,0,0) + 1(0,1,0) + 3(0,0,1) = (2,1,3)$

$T(0,0,1) = 1(1,0,0) + 1(0,1,0) + 4(0,0,1) = (1,1,4)$

Let $u = (x,y,z) \in R^3 \in U$

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

Taking Transformation on both sides

$$T(x,y,z) = x\,T(1,0,0) + y\,T(0,1,0) + z\,T(0,0,1)$$
$$= x(1,0,-1) + y(2,1,3) + z(1,1,4)$$

$$\boxed{T(x,y,z) = (x + 2y + z,\ y + z,\ -x + 3y + 4z)}$$ Ans

Scanned with CamScanner

Q.No.3 Find an orthonormal Basis of the inner product space $R^3(R)$ with the standard inner product, given the basis $\{(1,0,1),(0,1,1),(1,3,3)\}$ using Gram-Schmidt process. Also find the coefficient of the vector $(1,1,2)$ relative to the orthonormal Basis.

Soln Let $u=(x_1,x_2,x_3)$, $v=(y_1,y_2,y_3) \in R^3$

Standard inner product

$\langle u,v\rangle = (x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3)$

let $u_1=(1,0,1)$, $u_2=(0,1,1)$, $u_3=(1,3,3)$

$$\boxed{v_1 = u_1 = (1,0,1)}$$

$\|v_1\|^2 = \langle v_1, v_1\rangle = 1^2+0^2+1^2 = 2$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle . v_1}{\|v_1\|^2}$$

$$= u_2 - \frac{\langle u_2, u_1\rangle u_1}{2}$$

$\langle u_2, u_1\rangle = 1\times 0 + 0\times 1 + 1\times 1 = 1$

$$= u_2 - \frac{1 \cdot u_1}{2}$$

$$= (0,1,1) - \frac{(1,0,1)}{2} = (0,1,1) - \left(\tfrac{1}{2},0,\tfrac{1}{2}\right)$$

$$= \left(-\tfrac{1}{2}, 1, \tfrac{1}{2}\right) = \tfrac{1}{2}(-1,2,1)$$

$$\|v_2\|^2 = \tfrac{1}{4}\left((-1)^2+2^2+1^2\right) = \tfrac{1}{4}(1+4+1) = \frac{\cancel{6}^{3}}{\cancel{4}_{2}} = \frac{3}{2}$$

$$\boxed{\|v_2\|^2 = \frac{3}{2}}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle v_1}{\|v_1\|} - \frac{\langle u_3, v_2\rangle v_2}{\|v_2\|}$$

$$= u_3 - \frac{\langle u_3, u_1\rangle u_1}{2} - \frac{\langle u_3, v_2\rangle v_2}{\frac{3}{2}}$$

Scanned with CamScanner

$$V_3 = u_3 - \frac{\langle u_3, u_1\rangle}{2} u_1 - \frac{2}{3}\langle u_3, v_2\rangle v_2$$

$$\langle u_3, u_1\rangle = 1+3 = 4$$

$$\langle u_3, v_2\rangle = \langle (1,3,3), \tfrac12(-1,2,1)\rangle$$

$$= 1(-\tfrac12) + 3\times 1 + 3\times\tfrac12$$

$$= \cancel{-\tfrac12 + 6 + 3 = -\tfrac12 + 9 = \tfrac{17}{2}}$$

$$= -\tfrac12 + 3 + \tfrac32$$

$$\frac{-1+6+3}{2} = \frac82 = 4$$

$$V_3 = (1,3,3) - \frac{4}{2}(1,0,1) - \frac{2}{3}\times 4\times\frac12(-1,2,1)$$

$$= (1,3,3) - (2,0,2) - \frac43(-1,2,1)$$

$$= (1,3,3) + (-2,0,-2) + \left(\frac43, -\frac83, -\frac43\right)$$

$$= \left(\frac13, \frac13, -\frac13\right) = \frac13(1,1,-1)$$

$$\|V_3\|^2 = \frac19(1+1+1) = \frac{3}{9} = \frac13$$

$$W_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,1)}{\sqrt2} = \left(\frac1{\sqrt2}, 0, \frac1{\sqrt2}\right)$$

$$W_2 = \frac{v_2}{\|v_2\|} = \frac{\frac12(-1,2,1)}{\frac{\sqrt3}{\sqrt2}} = (-1,2,1)\cdot\frac12\times\frac{\sqrt2}{\sqrt3}$$

$$W_2 = (-1,2,1)\left(\frac1{\sqrt6}\right) = \left(-\frac1{\sqrt6}, \frac2{\sqrt6}, \frac1{\sqrt6}\right)$$

$$W_3 = \frac{v_3}{\|v_3\|} = \frac{\frac13(1,1,-1)}{\frac1{\sqrt3}} = (1,1,-1)\frac13\times\frac{\sqrt3}{1} = \frac{(1,1,-1)}{\sqrt3}$$

$$W_3 = \left(\frac1{\sqrt3}, \frac1{\sqrt3}, -\frac1{\sqrt3}\right)$$

Required orthonormal Basis $= \{w_1, w_2, w_3\}$

$$= \left\{\left(\frac1{\sqrt2}, 0, \frac1{\sqrt2}\right), \left(-\frac1{\sqrt6}, \frac2{\sqrt6}, \frac1{\sqrt6}\right), \left(\frac1{\sqrt3}, \frac1{\sqrt3}, -\frac1{\sqrt3}\right)\right\}$$

now find coefficient of (1, 1, 2) Relative to orthonormal Basis

$$(1,1,2) = a\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right) + b\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right) + c\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)$$

$$1 = \frac{a}{\sqrt2} - \frac{b}{\sqrt6} + \frac{c}{\sqrt3} \quad -(1)$$

$$1 = \frac{2b}{\sqrt6} + \frac{c}{\sqrt3} \quad -(2)$$

$$2 = \frac{a}{\sqrt2} + \frac{b}{\sqrt6} - \frac{c}{\sqrt3} \quad -(3)$$

equ. (1) + equ (3)

$$3 = \frac{2a}{\sqrt2} \Rightarrow a = \frac{3\sqrt2}{2} = \frac{3\cancel{\sqrt2}}{\sqrt2\times\cancel{\sqrt2}} = \frac{3}{\sqrt2}$$

$$\boxed{a = \frac{3}{\sqrt2}}$$ Put in (1)

$$1 = \frac{3}{\sqrt2}\times\frac{1}{\sqrt2} - \frac{b}{\sqrt6} + \frac{c}{\sqrt3}$$

$$1 - \frac{3}{2} = -\frac{b}{\sqrt6} + \frac{c}{\sqrt3} \Rightarrow -\frac{1}{2} = -\frac{b}{\sqrt6} + \frac{c}{\sqrt3} \quad -(4)$$

equ (2) − equ (4)

$$1 = \frac{2b}{\sqrt6} + \cancel{\frac{c}{\sqrt3}}$$

$$-\frac{1}{2} = -\frac{b}{\sqrt6} + \cancel{\frac{c}{\sqrt3}}$$

$$+ \qquad + \qquad -$$

$$\frac{3}{2} = \frac{3b}{\sqrt6} \Rightarrow b = \frac{\sqrt6}{2} = \frac{\sqrt3\times\cancel{\sqrt2}}{\sqrt2\times\cancel{\sqrt2}} = \frac{\sqrt3}{\sqrt2}$$

$$\boxed{b = \frac{\sqrt3}{\sqrt2}}$$ Put it into (2)

$$1 = \frac{2}{\sqrt6}\times\frac{\sqrt3}{\sqrt2} + \frac{c}{\sqrt3} \Rightarrow 1 = \frac{2\cancel{\sqrt3}}{\cancel{\sqrt3}\times\sqrt2\times\sqrt2} + \frac{c}{\sqrt3}$$

$$0 = \frac{c}{\sqrt3} \Rightarrow \boxed{c = 0}$$

Hence $$\boxed{(1,1,2) = \frac{3}{\sqrt2}\left(\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}\right) + \frac{\sqrt3}{\sqrt2}\left(-\frac{1}{\sqrt6}, \frac{2}{\sqrt6}, \frac{1}{\sqrt6}\right) + 0\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, -\frac{1}{\sqrt3}\right)}$$

A

Q.No.8 Let $T: R^3 \to R^2$ be a linear map defined by $T(x,y,z) = (3x+2y-4z, x-5y+3z)$ Find the matrix of T in the following basis R^3 & R^2 where

$B = \{(1,1,1), (1,1,0), (1,0,0)\}$ &

$B' = \{(1,3), (2,5)\}$.

Solu.

$T(1,1,1) = a(1,3) + b(2,5)$

$\Downarrow$

$$\boxed{(1,-1) = a(1,3) + b(2,5)}$$

$1 = a+2b$ —(i)

$-1 = 3a+5b$ —(ii)

eqn (i) X 3 − eqn (ii)

$3 = \cancel{3a} + 6b$

$-1 = \cancel{3a} + 5b$

$\underline{\quad - \quad -}$

$\boxed{4 = b}$ Put in ① $1 = a + 2 \times 4 \Rightarrow \boxed{a = -7}$

$T(1,1,0) = c(1,3) + d(2,5)$

$$\boxed{(5,-4) = c(1,3) + d(2,5)}$$

$5 = c+2d$ —(i)

$-4 = 3c+5d$ —(ii)

eq. (i) X3 − eq. (ii)

$15 = \cancel{3c} + 6d$

$-4 = \cancel{3c} + 5d$

$\underline{+ \quad -}$

$\boxed{19 = d}$ Put d into (i) $5 = c + 2 \times 19$

$$\boxed{c = 5 - 38 = -33}$$

$T(1,0,0) = e(1,3) + f(2,5)$

$$\boxed{(3,1) = e(1,3) + f(2,5)}$$

$3 = e+2f$ —(i)

$1 = 3e+5f$ —(ii)

eq. (i) X3 − eq. (ii)

$9 = \cancel{3e} + 6f$

$1 = \cancel{3e} + 5f$

$\underline{- \quad -}$

$\boxed{8 = f}$

Put the value of f in eq (i)

$3 = e + 2 \times 8$

$\boxed{e = 3 - 16 = -13}$

So

$T(1,1,1) = -7(1,3) + 4(2,5)$

$T(1,1,0) = -33(1,3) + 19(2,5)$

$T(1,0,0) = -13(1,3) + 8(2,5)$

$$\text{So } [T : B, B'] = \begin{bmatrix} -7 & 4 \\ -33 & 19 \\ -13 & 8 \end{bmatrix}'$$

$$= \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix} \text{ Ans}$$

<u>Q.No.9</u> Let W be a subspace of R^4 spanned by the vectors $u_1 = (1, -2, 5, -3)$, $u_2 = (2, 3, 1, -4)$, $u_3 = (3, 8, -3, -5)$

(i) Find Basis and dim of W.

(ii) Extended the basis of W to a basis of R^4

<u>Sol:</u> Basis of subspace W generated by the given vectors is obtained by reducing the matrix

$$A = \begin{bmatrix} 3 & 8 & -3 & -5 \\ 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \end{bmatrix} \text{ to row-echelon form}$$

$$\overset{R_{1,2}}{\sim} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 3 & 8 & -3 & -5 \\ 2 & 3 & 1 & -4 \end{bmatrix}$$

$$\sim R_{2,3} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero rows $(1,-2,5,-3)$, $(0,7,-9,2)$ form a basis of W.

Basis of $W = \{(1,-2,5,-3), (0,7,-9,2)\}$

$\dim W = 2$ Ans

Now to extend this basis to get a basis of $\mathbb{R}^4$, we need two more vectors, as $\dim \mathbb{R}^4 = 4$.

These vectors along with those of W will form a basis of $\mathbb{R}^4$

Let us consider the two other vectors are

$(0,0,1,0)$, $(0,0,0,1)$

obviously $\{(1,-2,5,-3), (0,7,-9,2), (0,0,1,0), (0,0,0,1)\}$ is a L.I. set.

as $A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a Row-echelon matrix

$\rho(A) = 4 =$ No. of column so L.I.

Hence basis of $\mathbb{R}^4 = \{(1,-2,5,-3), (0,7,-9,2), (0,0,1,0), (0,0,0,1)\}$ Ans